Fourier Optics in Examples

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This short lecture note presents some two-dimensional optical structures and their calculated Fourier transforms. These can be regarded as the respective far-field diffraction patterns. As an addition to textbooks, it may present some visual help to students working in the field of optics.

1 Paraxial Approximation

When light is propagating (here in positive z-direction), the electric field in an arbitrary plane at z can be calculated from the field at any preceding plane at z_0 applying Huygens's construction.

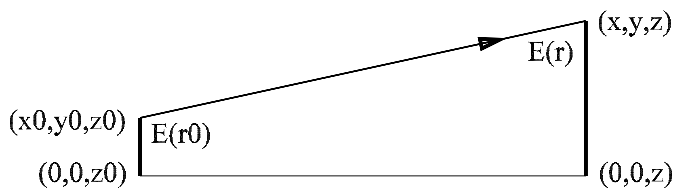

Figure 1: Geometry and parameters used in Eq. 1 for the paraxial approximation. Light is propagating from left to right ($z > z_0$).

The geometry is sketched in Fig: 1, for the field at $r = (x, y, z)$ contributed by the point $r_0 = (x_0, y_0, z_0)$ one may derive

$$E(r)_{r_0} \propto \frac{E(r_0)}{|r - r_0|} \exp(jk|r - r_0|) \quad , \tag{1}$$

assuming monochromatic, coherent light. Furthermore, we assume scalar E, which means that we consider only one polarization component and light propagation approximately parallel to the z-axis.

To get the total field, we have to integrate over x_0 and y_0 in the z_0-plane

$$E(r) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{E(r_0)}{|r - r_0|} \exp(jk|r - r_0|) dx_0 dy_0 \quad . \tag{2}$$

Here

$$|r - r_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{3}$$

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can be approximated if we assume that

$$(z-z_0)^2 \gg (x-x_0)^2 + (y-y_0)^2 \tag{4}$$

– this is called *Paraxial Approximation* – to

$$|r-r_0| = (z-z_0)\sqrt{1+\frac{(x-x_0)^2+(y-y_0)^2}{(z-z_0)^2}} \tag{5}$$

$$\approx (z-z_0)\left[1+\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)^2}\right] \quad . \tag{6}$$

Thus, using

$$\exp(jk|r-r_0|) = \exp(jk(z-z_0))\exp\left[jk\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)}\right] \tag{7}$$

and

$$\exp\left[jk\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)}\right] = \exp\left[jk\frac{x^2+y^2}{2(z-z_0)}\right]\exp\left[-jk\frac{xx_0+yy_0}{z-z_0}\right]\exp\left[jk\frac{x_0^2+y_0^2}{2(z-z_0)}\right] \quad , \tag{8}$$

Eq. 2 can be written as

$$E(x,y) \propto \frac{P(x,y)}{z-z_0}\iint E(x_0,y_0)P(x_0,y_0)\exp\left[-jk\frac{xx_0+yy_0}{z-z_0}\right]dx_0dy_0 \tag{9}$$

where

$$P(x,y) = \exp\left[jk\frac{x^2+y^2}{2(z-z_0)}\right] \quad . \tag{10}$$

Eq. 9 can be interpreted as a sequence of three operations:

- The field $E(x_0, y_0)$ is multiplied by a phase factor $P(x_0, y_0)$.
- For this product a two-dimensional Fourier transform is calculated.
- The result is multiplied by a second phase factor $P(x, y)$.

If the approximations leading to Eq. 9 can be made, this is called *Fresnel Diffraction* or *Fresnel Approximation* (for a more detailed introduction together with some sample calculations see e. g. [1]).

If, in addition, we can assume that $P(x_0, y_0) \approx 1$ in the entire region considered, i e. that $z - z_0$ is large enough, Eq. 9 can be rewritten as

$$E(x,y) \propto \frac{P(x,y)}{z-z_0}\iint E(x_0,y_0)\exp\left[-jk\frac{xx_0+yy_0}{z-z_0}\right]dx_0dy_0 \quad . \tag{11}$$

In this regime, $E(x, y)$ is just the two-dimensional Fourier transform of $E(x_0, y_0)$ except for a multiplicative phase factor which does not affect the intensity of the light. This regime is called *Fraunhofer Diffraction* or *Fraunhofer Approximation*. The examples presented here are calculated numerically assuming Fraunhofer approximation, i. e. simple two-dimensional Fourier transform.

2 Basic Features

First some basic features of the Fourier transform in two dimensions are outlined.

2.1 Dimensionality

When the two-dimensional pattern is only structured in one dimension, that also shows up in the Fourier transform, yet in a reciprocal meaning. This is visualized by Figs. 2 and 3.

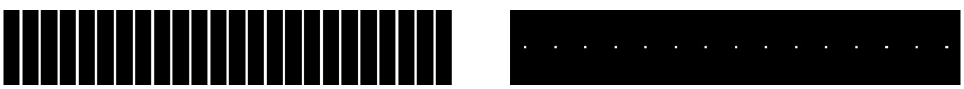

Figure 2: Array of lines (left) and the corresponding two-dimensional Fourier transform (right).

Figure 3: Array of points (left) and the corresponding two-dimensional Fourier transform (right).

In Fig. 2 the pattern is constant in the vertical dimension, its Fourier transform shows a delta function behavior in this dimension, yielding a linear array of points. Vice versa for Fig. 3. That's due to the fact that the Fourier transform of a constant is the delta function and vice versa.

2.2 Number of Elements

The number of elements in the original pattern strongly determines the sharpness of the diffraction pattern. Fig. 4 demonstrates this using a one-dimensional regular structure of points as source pattern. Depending on the number of points used, the diffraction pattern varies in sharpness.

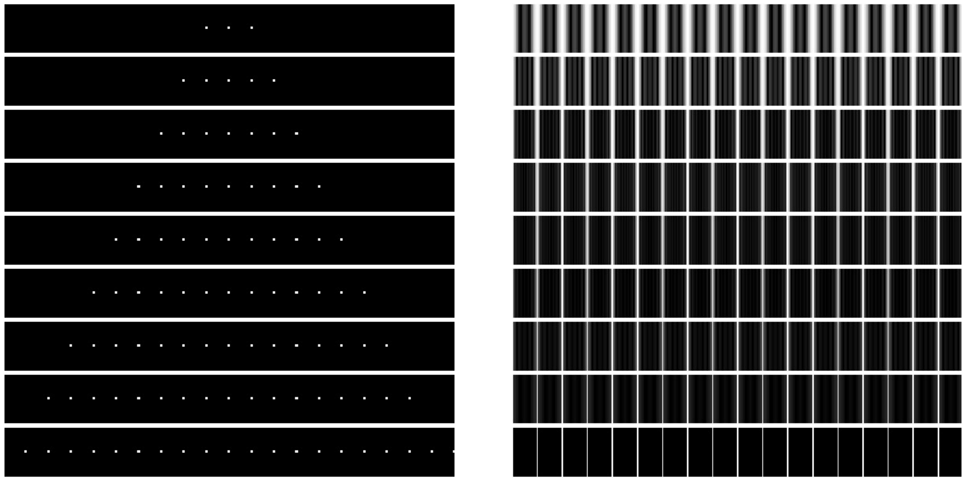

Figure 4: Dependence of the diffraction pattern on the number of source objects used. The sharpness increases with this number (from top to bottom: 3, 5, 7, 9, 11, 13, 15, 17, 19).

2.3 Apodization

A special problem in Fourier transform is the fact that one always has to deal with limited data, albeit theory assumes unlimited data to be transformed. Using limited data means to make a transformation of the product of the unlimited data with a rectangular function. The Fourier transform in that case is the convolution of the two transforms. As the transform of a rectangular function shows expressed side wings, these also show up in the transform of the product, mainly convoluted to each of the peaks of the transform. This spurious additional intensity may affect the pattern of the Fourier transform producing fictitious information. The effect is shown in Fig. 5, the peaks are smeared out, surrounded by undesired side wings.

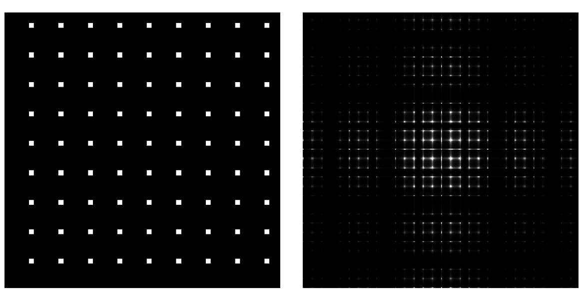

Figure 5: Sharply limited pattern (left) and its Fourier transform (right). Side wings appear at the peaks.

The effect can by reduced by smoothing the sharp edges of the original pattern. This treatment is called *apodization* or *windowing*. The pattern is multiplied by an appropriate *apodization* or *windowing* function. In Fig. 6 this is done using $\sin^2$

functions,

$$\text{pattern} = \text{pattern} \cdot \sin^2\left(\frac{x\pi}{s_x}\right) \cdot \sin^2\left(\frac{y\pi}{s_y}\right) \quad . \tag{12}$$

s_x and s_y, respectively, are the sizes of the original pattern.

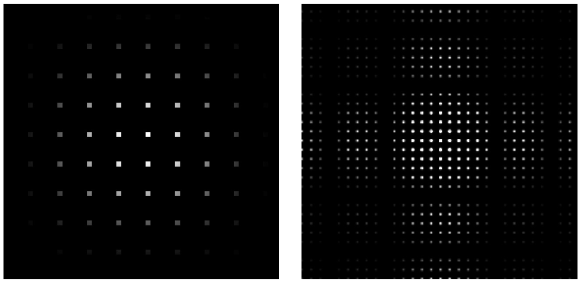

Figure 6: The pattern of Fig. 5 multiplied by an *apodization* function. The side wings in the Fourier transform disappear.

The resulting Fourier transform now is free from any side wings, yet the peaks are are broadened slightly. This is due to the suppression of the outer parts of the pattern which corresponds to an overall size and thus information reduction.

2.4 Aliasing

According to the *Sampling Theorem* the sampling frequency in discrete Fourier transform (DFT) must be at least twice the highest frequency to be detected. If this requirement cannot be met, *Aliasing* occurs, i. e. frequencies above this limit are aliased by corresponding lower frequencies. The principle is shown in Fig.7, a sinusoidal signal is sampled with too few sampling points, a lower frequency is pretended.

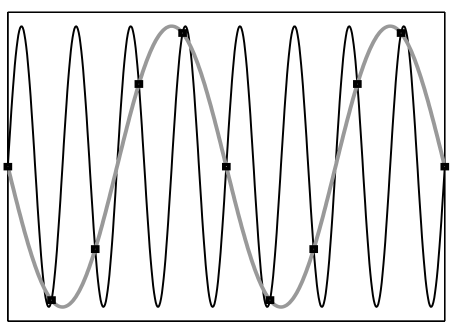

Figure 7: Aliasing: A sinusoidal signal (black) is sampled at the points indicated. From the sampled values, a lower frequency must be assumed (gray).

Similar results can occur in the simulation of optics using discrete Fourier transform. A diffraction pattern can thus be greatly distorted. Fig.8 shows an example: due to undersampling the number of minor maxima in the diffraction pattern is reduced.

Knowing this effect, it can be used 'beautify' the calculated diffraction patterns. Fig. 9 shows the way how to do it. In the left picture the samples are adjusted to

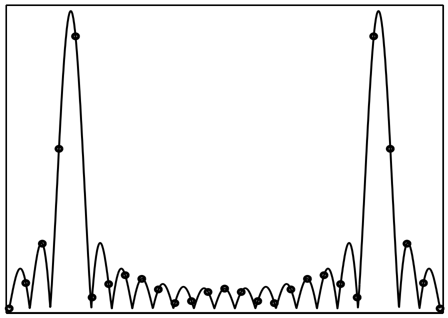

Figure 8: Undersampling of a diffraction pattern: Only 3 minor maxima instead of 13 seem to be present.

get a background-free diffraction pattern, in the right picture the number of samples is doubled to get an exact representation of the pattern.

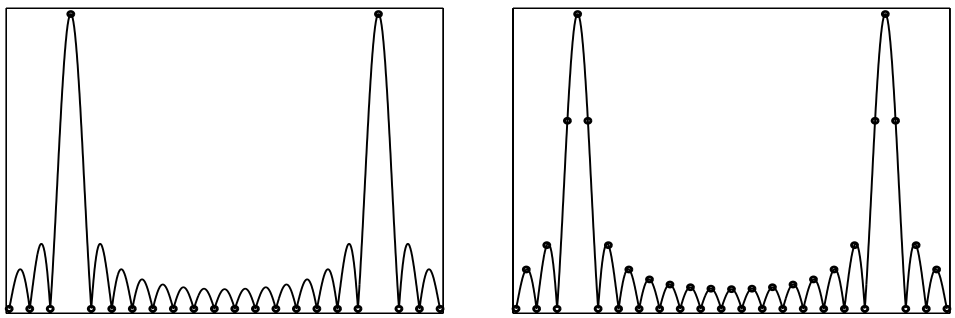

Figure 9: 'Background-free' Fourier transform of a grating structure using an appropriately adjusted sample density (left) compared to a reasonable representation of the structure using doubled sample density (right).

It can be calculated that we get the background-free representation when the total size of the pattern to be transformed equates exactly a multiple of the period. Fig: 10 shows this situation, the total size is 360, the period of the structure is 12 points. The Fourier transform of the grating structure is sharp and free of any minor maxima.

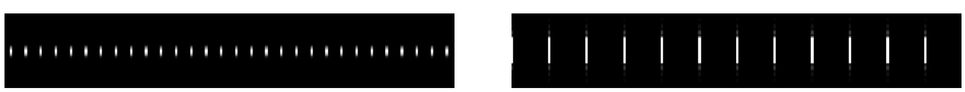

Figure 10: Grating structure with a multiple-period size. The Fourier transform is sharp and background-free.

If we don't meet this condition exactly, spurious intensity between the main peaks is produced. In Fig. 11 a period size of 13 with a total size of 360 points is used to visualize this.

The total number of samples in the Fourier transform equates the number of points in the original. The way to get a higher density of samples without adding more diffracting elements (which would also refine the diffraction pattern) is called *Zero*

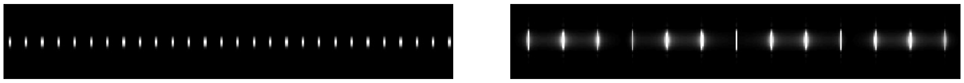

Figure 11: Grating structure with a size not equal to a multiple of the period (left), and corresponding Fourier tranform (right).

Padding. In the original an appropriate number of zeros is added to get the desired number of samples. To double the sample number, e. g., a zero matrix with the same size as the original has to be joined (Fig. 12).

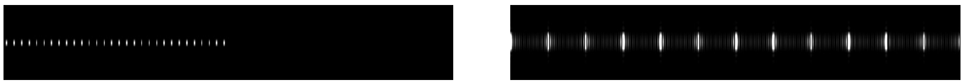

Figure 12: Zero padding: Grating structure of Fig. 10 padded with a zero matrix of the same size. In the Fourier transform the minor structures of the pattern are now visible.

2.5 Element Size

The extent of the diffraction pattern is complementary to the size of the single diffracting elements. Fig. 13 shows this reciprocal behavior using a single circular shape as example.

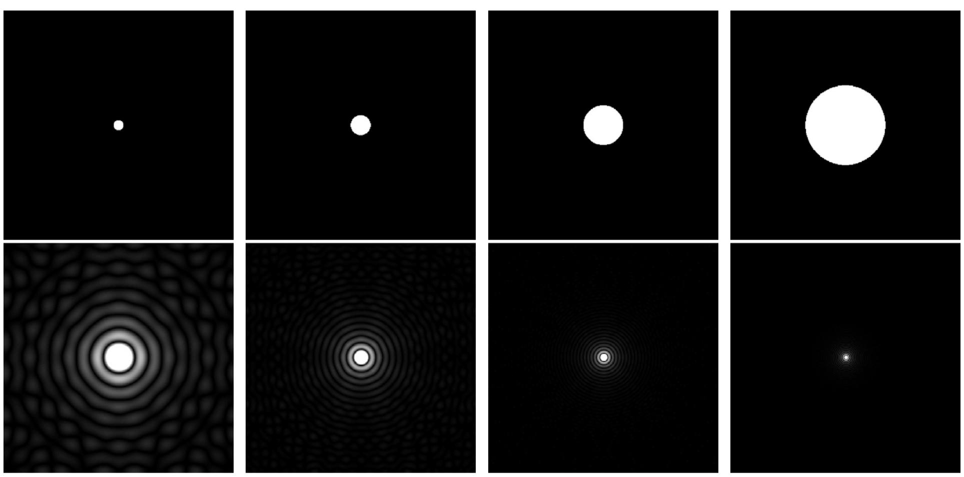

Figure 13: Circular apertures of different size (upper row) and their corresponding Fourier transforms (lower row). The intensities are normalized to their respective maximum value.

Similar results are produced by a two-dimensional regular array of objects. This is shown in Fig. 14, the extent of the diffraction patterns is reciprocal to the size of the single element.

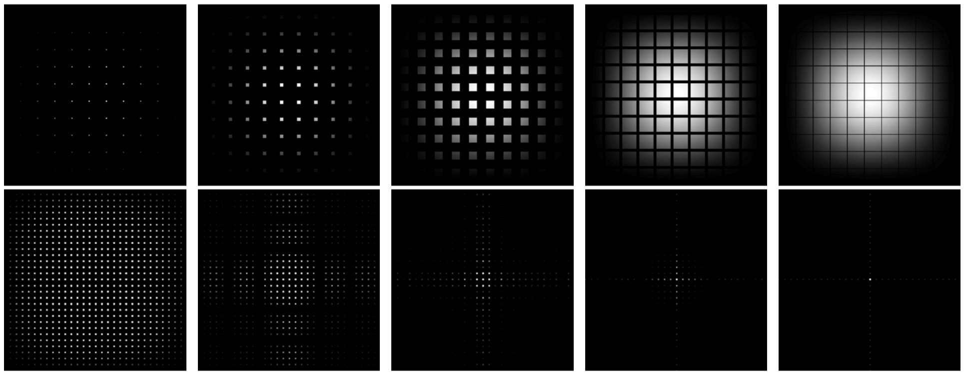

Figure 14: Arrays of square apertures of different size (upper row) and their corresponding Fourier transforms (lower row). The periodicity is kept constant.

2.6 Periodicity

A similar reciprocity as for the element size and the extent of the diffraction pattern of course must be valid for the periodicity of original and transform. This is one of the essentials of the Fourier transform. Fig. 15 demonstrates this behavior. Identical square sized objects are arranged with various periodicities.

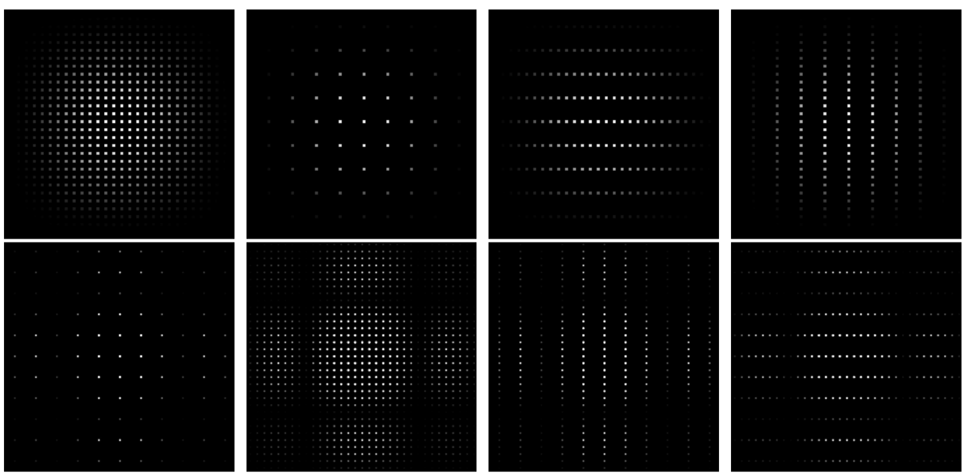

Figure 15: Identical objects arranged with different spacings between the single objects (upper row) and their corresponding Fourier transforms (lower row).

3 Single Element and Regular Array

In Fourier transform convolution and product are complementary mathematical operations. The Fourier transform of a product of two functions equates the convolution of the Fourier transforms of the two functions. Vice versa, the Fourier transform of a convolution of two functions equates the product of the two Fourier transforms of the single functions.

A regular array of identical elements can be treated as a convolution of an array of corresponding points and a single element. The Fourier transform then must equate the product of the two elementary transforms. Translated to diffraction optics this means that the diffraction pattern of a regular array can be calculated as the product of the diffraction pattern of a single element and the interference pattern of the point array.

Figs. 16 – 18 visualize this property of the Fourier transform.

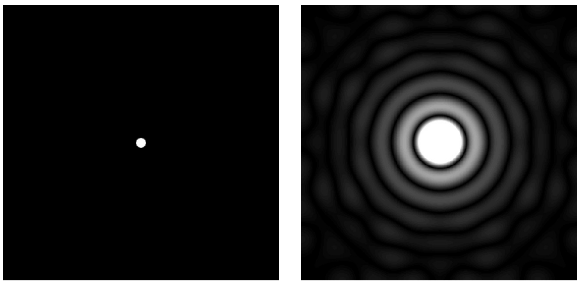

Figure 16: Single circular aperture and its Fourier transform.

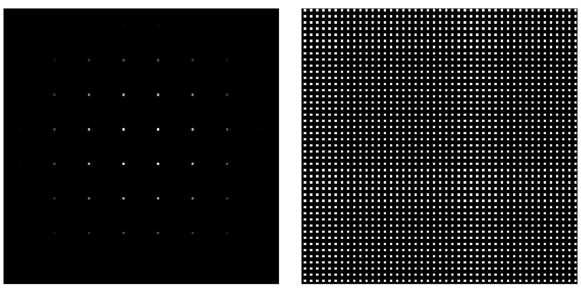

Figure 17: Regular array of points and corresponding Fourier transform.

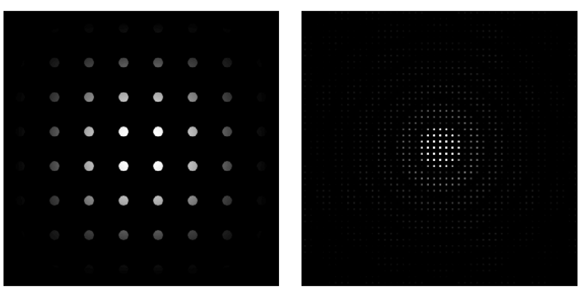

Figure 18: Regular array of circular apertures (*convolution* of single aperture and point array) and its corresponding Fourier transform (*product* of the respective Fourier transforms).

4 Gratings

A diffraction grating is a (one-dimensional) array of identical slits or mirror elements. The diffraction pattern can be calculated by two-dimensional Fourier transform in a similar way as discussed if we can assume *Fraunhofer Approximation*. Various typical features are shown in the following figures (appropriate aliasing is used to get sharp diffraction patterns – see Sec. 2.4).

Figure 19: Ideal grating (narrow slits) and corresponding diffraction pattern.

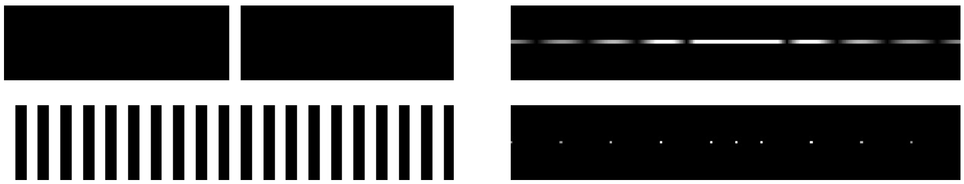

Figure 20: Slit width equates one half of the period: The minima of the slit diffraction function (top right) correspond to the maxima $N = \pm 2, \pm 4, \ldots$ of the grating diffraction resulting in missing maxima (bottom right).

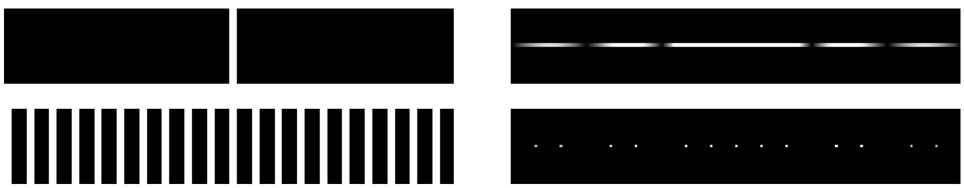

Figure 21: Slit width equates one third of the period: The minima of the slit diffraction function (top right) correspond to the maxima $N = \pm 3, \pm 6, \ldots$ (result bottom right).

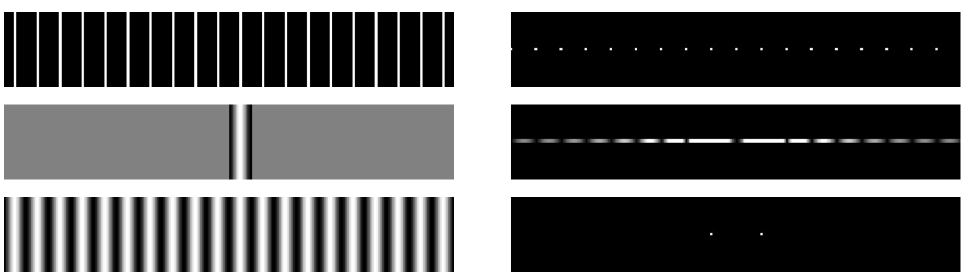

Figure 22: Grating with sinusoidal absorption pattern. A comparison of the ideal grating diffraction (top) with the diffraction pattern of a sinusoidal slit (middle) shows that all maxima except of those with $N = \pm 1$ coincide with minima of the slit function. This explains the resulting diffraction of a sinusoidal grating (bootom).

5 Random Patterns

Hitherto we dealt with regular patterns of identical objects. If such objects are irregularly distributed, different diffraction patterns are found. In these diffraction patterns, the form of the objects still can be detected, but no impression of any ordered structure. Two examples may visualize this.

Fig. 23 shows randomly distributed rectangles. The basic diffraction pattern is fading with increasing number of objects when – as usual – the intensity is normalized to the maximum intensity in the diffraction pattern. The reason for this is that an increasing portion of the total intensity is no longer diffracted but collected in a 'central peak' due to the randomness of the diffraction (in the limit of a completely white source pattern no diffraction pattern would show up, only a strong central diffraction peak).

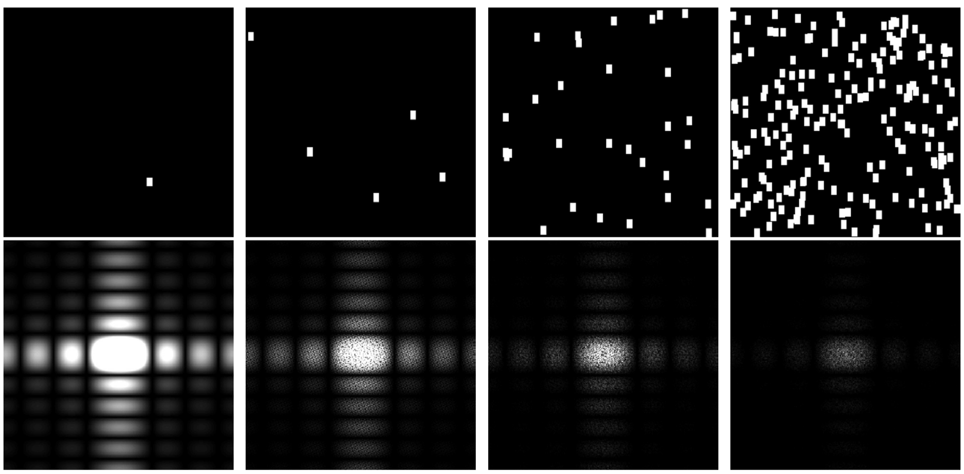

Figure 23: Randomly distributed identical objects (top row) and the corresponding diffraction pattern (bottom row). The number of objects is 1, 5, 30, and 200, respectively.

When the intensity scale is kept constant as shown in Fig. 24 a strong increase of the brightness is found instead.

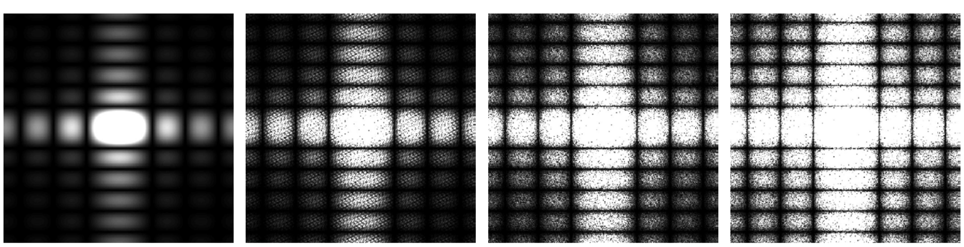

Figure 24: Diffraction pattern with constant intensity scale.

Good results are achieved with an intensity range proportional to the square root of the number of elements (Fig. 25) which obviously reflects the random behavior in the best way.

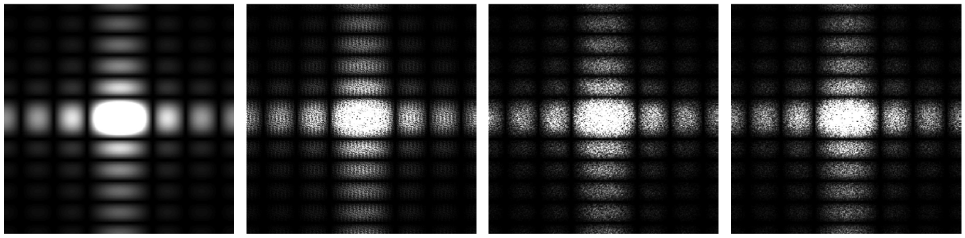

Figure 25: Intensity of the diffraction pattern scaled to the square root of the number of objects.

A second example is shown in Fig. 26, cross shaped small objects with equal orientation and size but random distribution. Even at a high number of objects, the basic diffraction pattern can be detected which e. g. allows for the determination of the size and of the orientation of the objects.

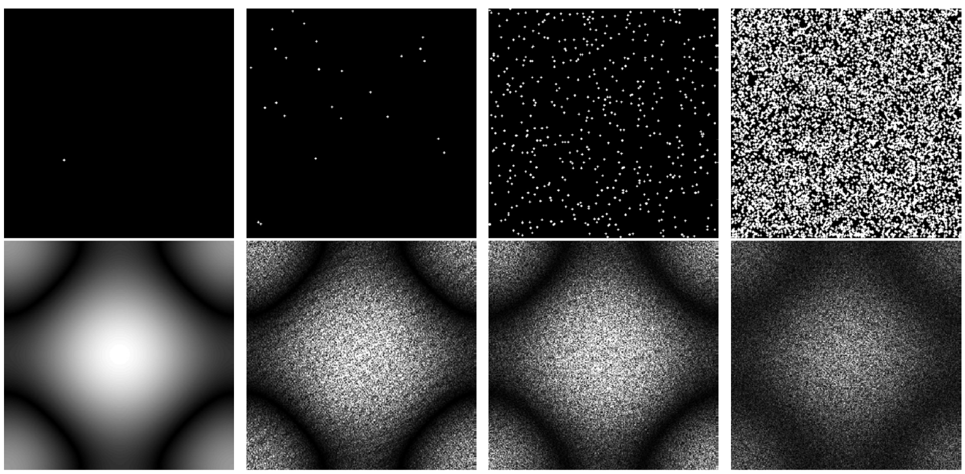

Figure 26: Randomly distributed small cross shaped objects (top row) and their diffraction pattern (bottom row). The number of objects is 1, 25, 500, and 10000, respectively. Intensity of the diffraction pattern is normalized with the square root of the number of objects.

References

[1] http://www.phy.auckland.ac.nz/Staff/smt/453701/chap7_00.pdf.